$U(n+1) \times U(p+1)$ - invariant Hermitian metrics with Hermitian tensor Ricci on the manifold $S^{2n+1} \times S^{2p+1}$.

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Abstract

Invariant complex structures on the homogeneous manifold $U(n+1)/U(n) \times U(p+1)/U(p)$ are reseached. The critical point of the functional of the scalar curvature is found.

Let consider the product $S^{2n+1} \times S^{2p+1}$ as homogeneous space $U(n+1)/U(n) \times U(p+1)/U(p)$. Suppose, that n and p are not equal to zero simultaneously. Denote $\mathfrak{g}_1$ and $\mathfrak{h}_1$ ($\mathfrak{g}_2$ and $\mathfrak{h}_2$) Lie algebras of Lie groups $U(n+1)$ and $U(n)$ ($U(p+1)$ and $U(p)$) respectively. As group $U(n)$ is embedded into $U(n+1)$ by usual way, then $\mathfrak{h}_j$ is embedded into $\mathfrak{g}_j$ by the following way:

$$M \in \mathfrak{h}_j \mapsto \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \in \mathfrak{g}_j,$$

where $j = 1, 2$. Let define the basis in $\mathfrak{g}_1 \times \mathfrak{g}_2$. Let $E^j_{\nu\mu}$ is matrix with 1 on the (ν, μ)-place and other zero elements. Define:

$$Z^j_{\nu\mu} = E^j_{\nu\mu} - E^j_{\mu\nu},\ T^j_{\nu\mu} = E^j_{\nu\mu} + E^j_{\mu\nu},\ 0 \le \mu < \nu \le n,\ j = 1, 2$$

Matrix $Z^j_{\nu\mu}$, $iT^j_{\nu\mu}$ (where $j = 1, 2$) form basis of product $\mathfrak{g}_1 \times \mathfrak{g}_2$. Take decomposition $\mathfrak{g}_j = \mathfrak{h}_j \oplus \mathfrak{p}_j$, where $\mathfrak{p}_j$ has basis $X^j = \frac{1}{2} i T^j_{00}$, $Y^j_{2\nu-1} = Z^j_{\nu 0}$, $Y^j_{2\nu} = iT^j_{\nu 0}$. So, manifold $S^{2n+1} \times S^{2p+1}$ viewed as homogeneous space $U(n+1)/U(n) \times U(p+1)/U(p)$ has basis $X^1, Y^1_{2\nu-1}, Y^1_{2\nu}, X^2, Y^2_{2\mu-1}, Y^2_{2\mu}$, $1 \le \nu \le n$, $1 \le \mu \le p$.

Proposition 1

$$\begin{array}{ll}
1)\ [\mathfrak{p}_0, \mathfrak{p}_1] \subset \mathfrak{p}_1 : & [X^1, Y^1_{2\nu-1}] = -Y^1_{2\nu}, [X^1, Y^1_{2\nu}] = Y^1_{2\nu-1}, \\
2)\ [\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{h}_1 \oplus \mathfrak{p}_0 : & [Y^1_{2\nu-1}, Y^1_{2\nu}] = -2X^1 + iT^1_{\nu\nu}, \\
& [Y^1_{2\nu}, Y^1_{2\mu}] = -Z^1_{\nu\mu}, \\
& [Y^1_{2\nu-1}, Y^1_{2\mu-1}] = -Z^1_{\nu\mu}, \\
& [Y^1_{2\nu}, Y^1_{2\mu-1}] = -T^1_{\nu\mu}, \\
3)\ [\mathfrak{p}_2, \mathfrak{p}_3] \subset \mathfrak{p}_3 : & [X^2, Y^2_{2\nu-1}] = -Y^2_{2\nu}, [X^2, Y^2_{2\nu}] = Y^2_{2\nu-1}, \\
4)\ [\mathfrak{p}_3, \mathfrak{p}_3] \subset \mathfrak{h}_2 \oplus \mathfrak{p}_2 : & [Y^2_{2\nu-1}, Y^2_{2\nu}] = -2X^2 + iT^2_{\nu\nu}, \\
& [Y^2_{2\nu}, Y^2_{2\mu}] = -Z^2_{\nu\mu}, \\
& [Y^2_{2\nu-1}, Y^2_{2\mu-1}] = -Z^2_{\nu\mu}, \\
& [Y^2_{2\nu}, Y^2_{2\mu-1}] = -T^2_{\nu\mu}, \\
5)\ [\mathfrak{p}_0 \oplus \mathfrak{p}_1, \mathfrak{p}_2 \oplus \mathfrak{p}_3] = 0 &
\end{array}$$

Proof. Proposition follows from definition of vectors $X^i, Y^i_{2\nu-1}, Y^i_{2\nu}$ $(i = 1, 2)$.

Definition 1 *Almost complex structure on the manifold M is smooth field of endomorphisms $J_x : T_xM \longrightarrow T_xM$, such that $J_x^2 = -Id_x$, $\forall x \in M$, where Id_x is identical endomorphism T_xM.*

Recall some known construction of complex structure on $S^{2n+1} \times S^{2p+1}$ [4]. It is known that $S^{2n+1} \times S^{2p+1}$ is principal $S^1 \times S^1$ bundle over $\mathbb{CP}^n \times \mathbb{CP}^p$. The space $\mathbb{CP}^n \times \mathbb{CP}^p$ and fiber $S^1 \times S^1$ are complex manifolds. If we fix complex structures on the base and fiber, then we can choose holomorphic transition functions to get complex structure on $S^{2n+1} \times S^{2p+1}$. All those structures form two parametric family $I(a, c)$ $(c > 0)$, they are $U(n + 1) \times U(p + 1)$ - invariant [4].

Consider projection:

$$U(n+1)/U(n) \times U(p+1)/U(p) \longrightarrow U(n+1)/(U(n) \times U(1)) \times U(p+1)/(U(p) \times U(1))$$

Obviously, that $U(n + 1)/(U(n) \times U(1)) \times U(p + 1)/(U(p) \times U(1))$ is product of complex projective spaces $\mathbb{CP}^n \times \mathbb{CP}^p$, and vectors X^1, X^2 are tangent to fiber. $I(a, c)$ acts on these vectors as

$$I(a,c)X^1 = \frac{a}{c}X^1 + \frac{1}{c}X^2, \quad I(a,c)X^2 = -\frac{a^2+c^2}{c}X^1 - \frac{a}{c}X^2$$

$$I(a,c)Y^1_{2\nu-1} = Y^1_{2\nu}, \quad I(a,c)Y^2_{2\mu-1} = Y^2_{2\mu},$$

where parameters a and c are real, $c > 0$. As $I(a, c)$ are $U(n + 1) \times U(p + 1)$ - invariant, then they defined by action of $I(a, c)$ on the basis of the space $\mathfrak{p}_1 \times \mathfrak{p}_2$. Denote $\mathfrak{p}_1 \times \mathfrak{p}_2$ as $\mathfrak{p}$, and $\mathfrak{h}_1 \times \mathfrak{h}_2$ as $\mathfrak{h}$.

Definition 2 *Almost complex structure J on the manifold M is called positive associated with 2-form ω, if:*
1) $\omega(JX, JY) = \omega(X, Y)$, for all $X, Y \in TM$
2) $\omega(X, JX) > 0$, for all nonzero $X \in TM$

Fix non-degenerate invariant 2-form ω:

$$\omega = X^1 \wedge X^2 + \sum_{\nu=1}^{n} Y^1_{2\nu-1} \wedge Y^1_{2\nu} + \sum_{\nu=1}^{p} Y^2_{2\nu-1} \wedge Y^2_{2\nu}$$

on $S^{2n+1} \times S^{2p+1}$

Lemma 1 *All complex structures $I(a, c)$ are positive associated with ω.*

Proof. For $I(a, c)$ properties 1) and 2) of definition 2 are obvious.

Corollary 1 *Every complex structure $I(a, c)$ defines unique ω - associated metric*

$$g(a, c)(X, Y) = \omega(X, I(a, c)Y)$$

These associated metrics are:

$$g(a, c)(X^1, X^1) = 1/c, \; g(a, c)(X^2, X^2) = (a^2 + c^2)/c, \; g(a, c)(X^1, X^2) = -a/c$$

$$g(a, c)(Y^1_j, Y^1_j) = g(a, c)(Y^2_k, Y^2_k) = 1, \; 1 \leq j \leq 2n, \; 1 \leq k \leq 2p$$

$$g(a, c)(X, Y) = 0, \; \text{for other basis vectors } X \text{ and } Y$$

Each metric of this family is $I(a, c)$ - Hermitian, so we obtain two-parametric family of Hermitian manifolds $(S^{2n+1} \times S^{2p+1}, \; g(a, c), I(a, c), \omega)$. Invariant metric induces scalar product on $\mathfrak{p}$.

Proposition 2 *Invariant Riemmanian connection for $g(a, c)$ on the $S^{2n+1} \times S^{2p+1}$ is given by formula $D_X Y = \frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y)$, where U is symmetric bilinear mapping $U : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathfrak{p}$:*

$$U(X^1, Y^1_{2\nu-1}) = \frac{2-c}{2c} \; Y^1_{2\nu}, \; U(X^1, Y^1_{2\nu}) = -\frac{2-c}{2c} \; Y^1_{2\nu-1},$$

$$U(X^1, Y^2_{2\nu-1}) = -\frac{a}{c} \; Y^2_{2\nu}, \; U(X^1, Y^2_{2\nu}) = \frac{a}{c} \; Y^2_{2\nu-1},$$

$$U(X^2, Y^1_{2\nu-1}) = -\frac{a}{c} \; Y^1_{2\nu}, \; U(X^2, Y^1_{2\nu}) = \frac{a}{c} \; Y^1_{2\nu-1},$$

$$U(X^2, Y^2_{2\nu-1}) = \left(\frac{a^2 + c^2}{c} - \frac{1}{2} \right) \; Y^2_{2\nu}, \; U(X^2, Y^2_{2\nu}) = -\left(\frac{a^2 + c^2}{c} - \frac{1}{2} \right) \; Y^2_{2\nu-1},$$

$U(X, Y) = 0$ for other basis vectors X and Y.

Proof. Find U by formula: $2g(U(X,Y),Z) = g([Z,X]_{\mathfrak{p}},Y) + g(X,[Z,Y]_{\mathfrak{p}})$

Proposition 3 *Two-parametric family of metrics $g(a,c)$ has following characteristics:*
1. Ricci curvature:

$$Ric(a,c)(X^1,X^1) = 2\frac{n+pa^2}{c^2},\ Ric(a,c)(X^2,X^2) = 2\frac{na^2+p(a^2+c^2)^2}{c^2},$$

$$Ric(a,c)(X^1,X^2) = -2\frac{a}{c^2}(n+p(a^2+c^2)),$$

$$Ric(a,c)(Y_j^1,Y_j^1) = 2(1+n-\frac{1}{c}),\ 1 \le j \le 2n,$$

$$Ric(a,c)(Y_k^2,Y_k^2) = 2(1+p-\frac{a^2+c^2}{c}),\ 1 \le k \le 2p,$$

$$Ric(a,c)(X,Y) = 0,\ \textit{for other basis vectors } X \textit{ and } Y$$

Proper values of Ricci curvature $\tilde{r}_i$ are $\tilde{r}_{1,2} = \frac{x+y\pm\sqrt{(x-y^2+4z^2)}}{2}$, where $x = 2\frac{n+pa^2}{c^2}$, $y = 2\frac{na^2+p(a^2+c^2)^2}{c^2}$, $z = -2\frac{a}{c^2}(n+p(a^2+c^2))$; $\tilde{r}_3 = \tilde{r}_4 = \cdots = \tilde{r}_{2n+2} = 2(1+n-\frac{1}{c})$, $\tilde{r}_{2n+3} = \tilde{r}_{2n+4} = \cdots = \tilde{r}_{2n+2p+2} = 2(1+p-\frac{a^2+c^2}{c})$.
2. Scalar curvature:

$$s = 4n\left(1+n-\frac{1}{2c}\right) + 4p\left(1+p-\frac{a^2+c^2}{2c}\right)$$

Proof.1. Calculate Ricci curvature by formula [1]:

$$Ric(X,X) = -\frac{1}{2}\sum_i |[X,Z_i]_{\mathfrak{p}}|^2 - \frac{1}{2}\sum_i g([X,[X,Z_i]_{\mathfrak{p}}]_{\mathfrak{p}},Z_i)$$

$$-\sum_i g([X,[X,Z_i]_{\mathfrak{h}}]_{\mathfrak{p}},Z_i) + \frac{1}{4}\sum_{i,j} g([Z_i,Z_j]_{\mathfrak{p}},X)^2 - g([Z,X]_{\mathfrak{p}},X),$$

where $Z = \sum_i U(Z_i,Z_i)$ and Z_i is orthonormal basis of the space $(\mathfrak{p},g)$.

In our case, orthonormal basis with respect to $g(a,c)$ is: $Z_0 = \sqrt{c}$, $Z_i = Y_i^1$, for $i = 1,\ldots 2n$, $Z_{2n+1} = \frac{a}{\sqrt{c}}X^1 + \frac{1}{\sqrt{c}}X^2$, $Z_{2n+1+i} = Y_i^2$, where $i = 1,\ldots,2p$. Obviously, that $Z = 0$. 2. Scalar curvature is calculated as trace of Ricci tensor: $s = Ric_{ij}g^{ij}$, where g^{ij} are components of $g(a,c)^{-1}$ $(i,j = 1,\ldots,2n+2p+2)$.

The family of complex structures $I(a,c)$ on $S^{2n+1}\times S^{2p+1}$ consists of all $U(n+1)\times U(p+1)$ - invariant almost complex structures. So, if $\mathcal{A}_\omega^+$ is space of invariant almost complex structures, which are positive associated with ω, and $\mathcal{AM}_\omega^+$ is the space of positive associated metrics, then:

$$\mathcal{A}_\omega^+ = \{I(a,c) : c > 0\} \qquad \mathcal{AM}_\omega^+ = \{g(a,c) : c > 0\}$$

The functional of scalar curvature is defined on the $\mathcal{AM}^+_\omega$:

$$s : \mathcal{AM}^+_\omega \longrightarrow \mathbb{R}, \qquad s(g) = 4n(1+n-\frac{1}{2c}) + 4p(1+p-\frac{a^2+c^2}{2c})$$

It is known (see, for example [2]), that critical points of this functional on $\mathcal{AM}^+_\omega$ give metrics with I- Hermitian Ricci tensor.

Proposition 4 *If n or p is equal to zero, then there are not $U(n+1)\times U(p+1)$ - invariant metrics $g(a,c)$ with Hermitian Ricci tensor on $S^{2n+1}\times S^{2p+1}$. If n and p are not equal to zero, then metric $g(a,c)$, when $a=0$, $c=\sqrt{\frac{n}{p}}$ has $I(a,c)$ - Hermitian Ricci tensor.*

Proof. Find partial derivatives of $s(a,c)$ with respect to a and c:

$$\frac{\partial s}{\partial a} = -p\frac{a}{c}$$

$$\frac{\partial s}{\partial c} = \frac{n-p(c^2-a^2)}{2c^2}$$

So, if n or p is equal to zero, then s has no critical points. If n and p are not equal to zero, then functional s takes maximal value $4n(n+1)+4p(1+p)-4\sqrt{np}$ at point $a=0$, $c=\sqrt{\frac{n}{p}}$.

Remark 1 *One can shows, that if $n=p$, $a=0$, $c=\sqrt{\frac{n}{p}}=1$, then:*

$$Ric = 2ng$$

Therefore the above metric is Einstein. If $n\neq p$, then above metrics are not Einstein.

Let $n\leq p$, we can apply result of [3]:

Proposition 5 *Sectional curvature of metric $g(0,\sqrt{\frac{n}{p}})$ satisfied to the following inequalities:*
1. If $0<\frac{n}{p}\leq\frac{1}{9}$, then $4-3\sqrt{\frac{p}{n}}\leq K\leq\sqrt{\frac{p}{n}}$. Minimal value is obtained on the bivector $Y^1_{2l-1}\wedge Y^1_{2l}$ $(l=1,\ldots,n)$, and maximal on the $\sqrt{c}X^1\wedge Y^1_i$ $(i=1,\ldots,2n)$.
2. If $\frac{1}{9}<\frac{n}{p}\leq\frac{9}{16}$, then $4-3\sqrt{\frac{p}{n}}\leq K\leq 4-3\sqrt{\frac{n}{p}}$. Minimal value is obtained on the bivector $Y^1_{2l-1}\wedge Y^1_{2l}$ $(l=1,\ldots,n)$, and maximal on the $Y^2_{2m-1}\wedge Y^2_{2m}$ $(m=1,\ldots,p)$.
3. If $\frac{9}{16}<\frac{n}{p}\leq 1$, then $0\leq K\leq 4-3\sqrt{\frac{n}{p}}$. Minimal value is obtained on bivectors $X^1\wedge X^2$, $Y^1_{2l-1}\wedge Y^2_{2m-1}$ and $Y^1_{2l}\wedge Y^2_{2m}$ $(l=1,\ldots,n,\ m=1,\ldots,p)$, maximal on the $Y^2_{2l-1}\wedge Y^2_{2l}$.

References

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